Twisted forms of linear algebraic groups: Relative root subgroups

Sergei Haller

Sergei.Haller@math.uni-giessen.de

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The Project

- Main goal: algorithms for computation in linear algebraic groups
- Joint with Arjeh M. Cohen, Scott H. Murray and Don E. Taylor
- Implementation in Magma

Preliminary remarks

- Classification of finite simple groups
  - Chevallley groups (“untwisted”)
  - Twisted Chevallley groups (unitary, Ree, Suzuki, . . .)

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  - Steinberg presentation
  - Unique decomposition of elements

Preliminary remarks

- Classification of finite simple groups
  - Chevallley groups (“untwisted”)
  - Twisted Chevallley groups (unitary, Ree, Suzuki, . . .)
- Computation in “untwisted” groups possible for arbitrary fields
  - Steinberg presentation
  - Unique decomposition of elements
- Computation in “twisted” groups is not possible
- More twisted groups for arbitrary fields

Linear Algebraic Groups

- F is an algebraically closed field

- Linear algebraic groups are
  - subgroups of GL_n for some n
  - defined by polynomial equations (over F)

- Examples:

$$\mathrm{GL}_n = \{ \ (A, t) \in F^{n \times n+1} \mid \det(A)t = 1 \ \}$$

$$\mathrm{SL}_n = \{ \ A \in F^{n \times n} \mid \det(A) = 1 \ \}$$

$$\mathrm{SU}_n = (\mathrm{SL}_n)_{\boldsymbol{\alpha}}$$

Field of Definition

- F is an algebraically closed field
- G is a linear algebraic group

- G is <u>defined over</u> the subfield $k \subseteq F$ if
  the polynomials involved in the definition of G are over k

- Examples:
  GL_n and SL_n are defined over the prime field of F

Groups of Lie type

- For a Galois extension $k \subseteq K \subseteq k_{sep}$ and $\Gamma = \mathrm{Gal}(k_{sep} : K)$

$$G(K) = \{\, g \in G \mid g^\gamma = g \quad \forall \gamma \in \Gamma \,\}$$

(here Γ acts componentwise on the entries)

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- G is a reductive linear algebraic group defined over k
  K is a Galois extension of k

$$\Gamma := \mathrm{Gal}(K : k) \qquad A := \mathrm{Aut}_K(G)$$

Twisted Groups

- For a cocycle $\boldsymbol{\alpha}$:

$$G_{\boldsymbol{\alpha}}(k) = \{\ g \in G(K) \mid g^{\gamma \boldsymbol{\alpha}_\gamma} = g \quad \forall \gamma \in \Gamma\ \} \leq G(K)$$

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- “Untwisted” groups of Lie type given by the trivial cocycle:

$$G_{\mathbf{1}}(k) = \{ \; g \in G(K) \mid g^{\gamma} = g \quad \forall \gamma \in \Gamma \; \} = G(k)$$

Problem description

- Which elements are inside $G_{\boldsymbol{\alpha}}(k)$?
  - easy to decide for a given element
  - hard to find new elements

Steinberg presentation and Root datum

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- Relations:

$$x_r(t)x_r(u) = x_r(t+u)$$

$$[x_r(t), x_s(u)] = \prod_{i,j>0} x_{ir+js}(C_{rsij}t^i u^j)$$

$$\dots$$

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Γ-action on the root system

- Each α_γ can be assumed to be of the form

$$\tau \dot{w} h$$

  - τ is Diagram automorphism
  - w is Weyl element
  - h is torus element

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- Ψ is called relative root system

Relative roots

- For a relative root $\delta \in \Psi^+$ we have

$$\pi^{-1}(\delta) = \dot{\bigcup_{r \in J_\delta}} \mathcal{O}_{\boldsymbol{\alpha}}(r) \subseteq \Phi^+ \setminus \Phi_0.$$

- Here J_δ is a fixed set of representatives of involved orbits

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- Two cases:

  - $2\delta \notin \Psi$
  - $2\delta \in \Psi$

Relative root elements: Case $2\delta \notin \Psi$

- In this case root elements are

$$x_\delta(t) := u_\delta(t)$$

- The root subgroup is abelian group

$$X_\delta = U_\delta$$

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- $c(t)$ is product of root elements corresponding to roots in $\pi^{-1}(2\delta)$

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- Use modified version of additive Hilbert's Theorem 90

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- Note that the definition of X_δ is independent of the choice of elements $v(t)$

Groups generated by relative roots

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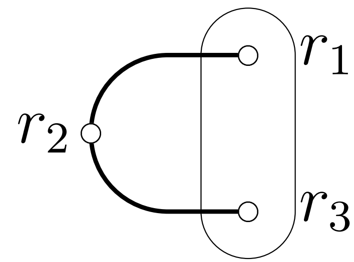
Example

$^2A_{3,1}(k)$



$$\mathcal{O}_{\boldsymbol{\alpha}}(r_1) = \{r_1, r_2 + r_3\}$$

$$\mathcal{O}_{\boldsymbol{\alpha}}(r_2) = \{r_2, -r_2\}$$

$$\mathcal{O}_{\boldsymbol{\alpha}}(r_3) = \{r_3, r_1 + r_2\}$$

$$\mathcal{O}_{\boldsymbol{\alpha}}(r_*) = \{r_*\}$$